Descriptive Set Theory
Lecture 1

The story. Cantor wanted to prove Continuum Hypothesis (CH).

$$\text{Continuum} := |\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}|$$

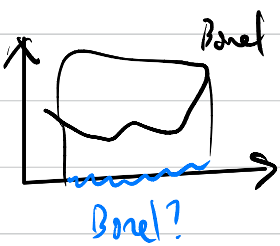
CH. There is no set A with $|\mathbb{N}| < |A| < |2^{\mathbb{N}}|$.

Cantor managed to prove a stronger property for all closed subsets of $\mathbb{R}$.

Perfect set property. PSP A subset $A \subseteq \mathbb{R}$ has the *perfect set property* if it's either ctbl or it is "perfect" (this implies $2^{\mathbb{N}} \hookrightarrow A$ continuous injection).

Thus, Cantor showed that closed sets have the PSP, hence also ctbl unions of closed sets (i.e. F_σ sets) do. He also showed (I think) that G_δ sets (i.e. ctbl intersections of open sets) have the PSP. He thought that if he "keeps going", he'll eventually show that all subsets of $\mathbb{R}$ have the PSP, so the CH holds.

However, it's is easy to get a set from the Axiom of Choice that doesn't satisfy the PSP. However, maybe PSP holds for all definable sets, i.e. sets that can be built from open sets via complements, ctbl unions/intersections, projections. The idea of this sets was made formal first by Borel, who defined Borel sets as all sets that can be obtained from open using ctbl unions and complements, i.e. the smallest σ-algebra containing open sets. Then Lebesgue defined measurable sets and measure on $\mathbb{R}$. He published a proof that projections of Borel sets in $\mathbb{R}^d$ are still Borel. About 10 years later, Souslin found a mistake in this proof, and moreover constructed an example of a G_δ subset of $\mathbb{R}^2$ whose projection to $\mathbb{R}$ is not Borel.



So, projections of Borel sets are new kinds of sets, which Luzin (Souslin's advisor) and Souslin called analytic, and started the systematic study of this sets and DST was born.

It was quickly proven that analytic sets are still Lebesgue measurable, hence so are co-analytic sets, i.e. complements of analytic sets. Souslin showed that proj (co-analytic) $\subsetneq$ analytic $\cup$ co-analytic, so again we have new sets. If you "keep going", you get the projective hierarchy of sets. Then these analysts were stuck on

Question. Are proj (co-analytic) Lebesgue measurable?

This remained open until logicians, namely set theorists, prove that this question is independent of ZFC. In fact, there is a concrete G_δ subset $A \subseteq \mathbb{R}^3$ s.t. whether $\mathrm{proj}(\mathrm{proj}(A)^c)$ is measurable is indep of ZFC. Then 60-80, DST was part of set theory and the questions were how independent statements about definable subsets of reals can be. Until mid-80s - early 90, Alexander Kechris and others revived the classical DST and demonstrated that DST is highly applicable in studying objects that arise in analysis (harmonic analysis), dynamics (ergodic theory, topological dynamics), and operator algebras (C^*-algebras, von Neumann algebras). Moreover, in 1995, Kechris, Solecki, Todorcevic gave rise to a new subject: descriptive graph combinatorics, which is currently booming. By the way, we mentioned $\mathbb{R}$ and $\mathbb{R}^d$ only because that's what analysts used to study, but really, DST works with Polish spaces. These are topological spaces that are completely metrizable and separable ($\iff$ 2nd ctbl).

Review of metric spaces and pointset topology.

A metric space (X, d) is complete if every Cauchy sequence in it converges.

Prop. For metric space X, TFAE.

(1) X is complete.

This is what we'll use. (2) Every decreasing sequence $(C_n)_{n \in \mathbb{N}}$ of closed sets with vanishing diameter [i.e. $\operatorname{diam}(C_n) \to 0$ as $n \to \infty$, where $\operatorname{diam}(C) := \sup\{d(x,y) : x, y \in C\}$] has a nonempty intersection. (Note: this intersection contains one point.)

(3) Every decreasing sequence of closed balls of vanishing diameter has an intersection.

Now let X be a topological space, i.e. it's really $(X, \mathcal{T})$, where $\mathcal{T}$ is the collection of open sets (i.e. $\emptyset, X \in \mathcal{T}$ and $\mathcal{T}$ is closed under all unions and finite intersections).
A metric space is a top. space with open sets = unions of open balls. A top. space is metrizable if it arises from a metric. An example of a non-metrizable space our society (or Facebook) with social circles as open sets.
A basis in a top. space is a collection $\mathcal{B}$ of open sets s.t. every nonempty open set is a union of sets from $\mathcal{B}$.
We say that a collection $\mathcal{C}$ of open sets generates the topology on X if the top on X is the smallest topology containing $\mathcal{C}$.

Prop. $\mathcal{C}$ generates the top on X $\Leftrightarrow$ the collection of all finite intersections of sets in $\mathcal{C}$ is a basis for X.

For example, in a metric space, balls form a basis.
A top. space is called 2^{nd} ctbl if it admits a ctbl basis. A subset $D \subseteq X$ of a top. space X is called dense if $\forall\ \emptyset \neq U \subseteq X$ open, $U \cap D \neq \emptyset$. A space is sepa-
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rable if it admits a ctbl dense subset.

<u>Observation.</u> 2^{nd} ctbl $\Rightarrow$ separable.

$\Leftarrow$ (for metric spaces: set of balls of rational radius about the points in a ctbl dense set).